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LETTER TO THE EDITOR

A precise determination of the backbone fractal dimension on two-dimensional percolation clusters

Mark D Rintoul and Hisao Nakanishi
Department of Physics, Purdue University, West Lafayette, IN 47907, USA

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Abstract. The backbone fractal dimension d_f^B is calculated on two-dimensional percolation clusters at the percolation threshold. Studies are carried out on backbones defined in three different ways: bus bar, point-to-point, and with fully periodic boundaries. The estimates of d_f^B are obtained by measuring the variation of mass with radius for all clusters, and by means of finite size scaling on the bus bar backbones. All cases imply a value of $d_f^B = 1.64 \pm 0.01$ for the backbone fractal dimension. Because of the high degree of self-consistency of all the results, we believe that this estimate represents a considerably improved accuracy.

Percolation clusters have been suggested as models in many physical processes [1, 2]. However, in many applications it is the backbone of the percolation cluster, rather than the percolation cluster itself, that contains the relevant physics. The backbone is usually defined for a finite system as the bonds which carry a DC electrical current between two points, or sets of points, kept at different voltages. Other physical realizations where the backbone may play an essential role include fluid flow in porous media, and the elasticity of gels. A more geometrical definition of the backbone is the union of all self-avoiding walks between two sets of points. This allows an extension of the definition of the backbone to infinite systems, where it is defined as the union of all *infinite self-avoiding walks originating from a given set of points.*

Extensive research has been done on the backbone to determine various quantities which characterize its structure. One of the most important of these quantities is its fractal dimensionality, d_f^B, which describes the distribution of mass surrounding any given point. There have been many previous simulations done to determine d_f^B for various dimensionalities and structures. Unfortunately, the results of these simulations have varied significantly from work to work. In two dimensions reported values for d_f^B range from 1.60 to 1.68, and this variation is even larger in higher dimensions [3–8]. In this letter, we propose to obtain a very accurate determination of d_f^B using three different backbone structures and two different methods.

The exponent d_f^B is just a measure of how the mass of a cluster backbone varies asymptotically with distance from a given point on a cluster and is defined by the following relationship

$$M(R) \propto R^{d_f^B}. \tag{1}$$

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This allows d_f^B to be determined by looking at the amount of mass contained in disks with different R centred about a random set of points in a cluster. The backbone fractal dimension can also be calculated from the scaling relation

$$d_f^B = d - \beta_B/\nu \tag{2}$$

where ν is the standard correlation length exponent (presumed to be $4/3$ in two dimensions) and β_B describes the total fraction of sites which belong to the backbone

$$P(p) \propto (p - p_c)^{\beta_B}. \tag{3}$$

Applying finite-size scaling to equation (3) we obtain

$$P(p, L) \propto |p - p_c|^{\beta_B} f(L|p - p_c|^{\nu}) \tag{4}$$

$$\propto L^{-\beta_B/\nu} g(|p - p_c|L^{1/\nu}). \tag{5}$$

Evaluating equation (5) at $p = p_c$ gives

$$P(p_c, L) \propto L^{-\beta_B/\nu} \tag{6}$$

where L is the length scale of the lattice containing the cluster. If $P(p_c, L)$ is calculated for various L a value of β_B/ν can be found and d_f^B can be determined. Similar methods were used to obtain d_f^B for the point-to-point backbone by Herrmann *et al* [3] and for the fully periodic backbone by Puech and Rammal [5], where they reported an accurate determination of d_f^B equal to 1.60 ± 0.05 and 1.68 ± 0.02, respectively. It should be noted that these two estimates do not have overlapping error bars. Since the standard static percolation exponents such as γ, β and ν are known exactly (see, e.g., [2]) for two dimensions, we should address the seeming indeterminacy for d_f^B.

The first backbone structure which we study is the bus bar backbone structure. This structure represents the current carrying bonds on a random resistor network on which two opposite edges are kept at different potentials. The bus bar backbones were created by first labelling a site on an $L \times L$ square lattice as either occupied or unoccupied based on a given probability. The largest cluster was then identified and accepted if it spanned from one side of the cluster to the opposite side. The bus bar boundary conditions were then applied and the dangling ends removed through a burning algorithm similar to that described by Herrmann *et al* [3]. A depth-first backbone extraction algorithm was also written to check that the burning algorithm was working properly.

The exponent d_f^B was first directly measured by the method using disks as described earlier. A measure of the mass as a function of radius was averaged over 100 randomly selected points throughout each disorder configuration and these averages were subsequently averaged over 1000 different configurations. The different radii used were taken in integer steps, until the edge of the original lattice was reached. The lattice size, L, ranged from 100 to 1000. The resulting graph of mass as a function of the disk radius, R, for $L = 1000$ is shown in figure 1. The slope remains remarkably constant for most values of R, as the inset shows. Only for large R does

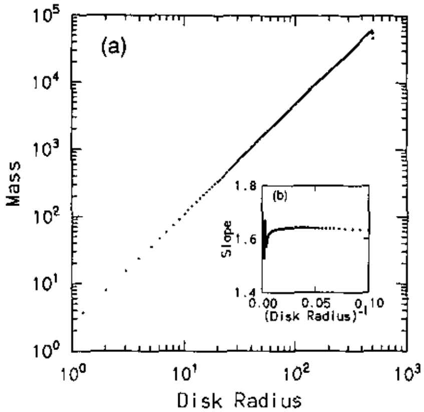

Figure 1. (*a*) Plot of mass against radius for bus bar backbones. These data were taken from the backbones of clusters generated on a 1000 x 1000 lattice. There were 1000 clusters sampled with 100 sample points taken on each cluster as disk centres. (*b*) The inset is a graph of the effective slope of the line in (*a*). This slope was calculated by including successively larger disk radii in the same way as in [9].

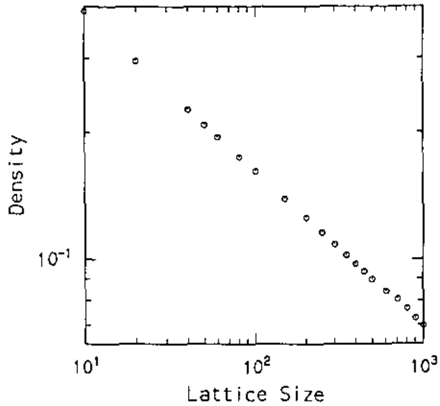

Figure 2. Plot of the average density of the bus bar backbones as a function of lattice size. The slope of this line should be equal to $-\beta_B/\nu$ and has a numerical value of 0.365 ± 0.005. This corresponds to a value of 1.635 ± 0.005 for d_f^B.

the slope decrease due to edge effects. The slope of the graph corresponds to a value of 1.64 ± 0.01. Similar graphs for different L show no significant differences.

The cluster density at $p = p_c \approx 0.59273$ is plotted against L in figure 2. For this data, we chose $100 \leqslant L \leqslant 1000$, and the number of clusters generated varied from 10^5 for $L = 100$, to 1000 for $L = 1000$. If finite size scaling laws apply, the slope of a line fit to the data should give a value of β_B/ν. For our data, we find $\beta_B/\nu = 0.365 \pm 0.005$, which corresponds to a value of 1.635 ± 0.005 for d_f^B. Figure 3 shows all of our data for $p \neq p_c$ collapsed into a single plot of $P(p, L)|p - p_c|^{-\beta_B}$ against $L|p - p_c|^{\nu}$. All of the points tend to fall on two lines, representing the two branches of $f(x)$ (as defined in equation (4)) for $p < p_c$ and $p > p_c$.

The second type of backbone that we studied was the point-to-point (PP) backbone. This backbone was generated by methods similar to that of the bus bar backbone, except that the bus bar boundary conditions were not applied. For this case, the two most diagonally opposite points were chosen from each edge, and the backbone between those two points was identified. Simulations were done for $100 \leqslant L \leqslant 1000$. These runs once again gave a value of 1.64 ± 0.01 for d_f^B, from the calculation of the slope of the mass against radius plot. The graph for $L = 1000$ is shown in figure 4. At large R the slope drops off much more rapidly with radius than the bus bar backbone does, which is to be expected because the 'stringier' nature of the backbone causes the finite size effect to become more pronounced.

The third type of backbone we looked at was the backbone defined on an infinitely periodic cluster. To construct this object, we first created a percolation cluster on an $L \times L$ lattice and identified the largest cluster. This cluster was accepted if it was

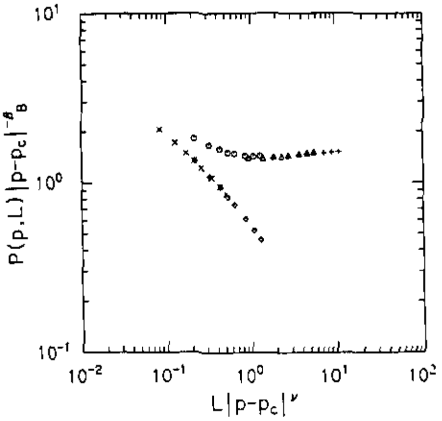

Figure 3. Plot of $P(p, L)|p-p_c|^{-\beta_B}$ as a function of $L|p-p_c|^\nu$ for all data where $p \neq p_c$. The two curves represent the two branches of $f(x)$ from equation (4), where $p < p_c$ for the lower branch and $p > p_c$ for the upper branch.

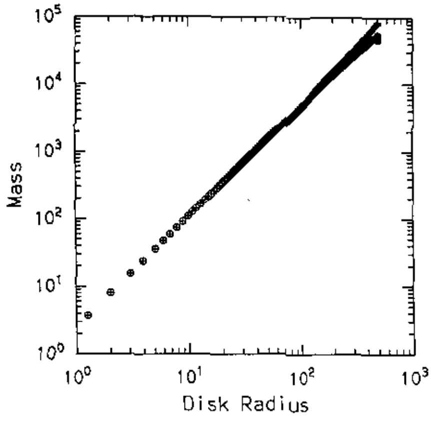

Figure 4. Plot of mass against radius for the point-to-point (⊙) and fully periodic (+) backbones. These data were taken from the backbones of clusters generated on a 1000×1000 lattice. There were 1000 clusters of each type sampled with 100 sample points taken on each cluster.

wrapping in both directions, otherwise it was rejected. Periodic boundary conditions were then imposed and other smaller clusters were then added if the boundary conditions caused them to become connected to the largest cluster. The dangling ends were then removed by a burning algorithm similar to the one used in the previous type of backbone. These data again yielded a value of 1.64 ± 0.01 for d_f^B, although the value of the exponent rose dramatically for large radii due to edge effects. These data are also shown in figure 4.

To summarize our results, we believe that the value of d_f^B in two dimensions is 1.64 ± 0.01. This value was obtained using three different types of backbones, and two very different methods for extracting the value of the exponent. This provided a self-consistency check for our result. This result is also consistent with other published results. The stringent internal consistency requirement it passes indicates that our estimate is indeed a rather accurate one.

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